Liouville field theory: IST and Poisson bracket structure

1986 J. Phys. A: Math. Gen. 19 121

(http://iopscience.iop.org/0305-4470/19/1/015)

J. Phys. A: Math. Gen. 19 (1986) 121-139. Printed in Great Britain

Liouville field theory: IST and Poisson bracket structure

G P Jorjadze†, A K Pogrebkov, M C Polivanov and S V Talalov
Steklov Mathematical Institute, Moscow, USSR

Received 12 March 1985

Abstract. The general solution for the Liouville equation with any given number N of singularities is considered. With the help of the inverse scattering transform method (IST) the set of regular continuous and discrete canonical variables is derived. The dynamical generators of Poincaré and dilatation groups and N-soliton solutions are constructed in terms of these variables.

1. Introduction

It is widely known that the straightforward application of the inverse scattering method to the Liouville equation

$$\varphi_{tt}-\varphi_{xx}+2\mathrm{e}^{\varphi}=0 \tag{1.1}$$

does not work. Indeed beginning, for example, with the standard Lax pair for the sine-Gordon equation we obtain the pair for the Liouville equation as a formal limit $\varphi\to\infty$ after omitting terms of $\mathrm{e}^{-\varphi}$ type. It is easily seen (Pogrebkov and Polivanov 1985) that the dependence of this pair on the spectral parameter in fact ceases. Attempts were made to overcome this difficulty by introducing some special asymptotic behaviour, as in Andreev (1976) and D'Hoker and Jackiw (1982). However, this leads to another difficulty (Jackiw 1984): the resulting theory becomes non-translation invariant. Another idea was introduced in Gervais and Neveu (1982) where the cone coordinates $x\pm t$ were considered as spectral parameters in some sense. This gives a somewhat restrictive picture which does not allow us to consider the case of singular solutions. In a modified form this approach was used by Faddeev and Takhtadjan (see Pogrebkov and Polivanov 1985) and proved to lead to complicated constrained dynamics in the singular case.

In our previous investigations of the Liouville equation (1.1) (see Dzordzhadze *et al* 1979, Pogrebkov and Polivanov 1985) we did not use the inverse scattering method but expressed the Liouville solution in terms of two arbitrary functions $A_\pm(x_\pm)$

$$\varphi(t,x)=\log\frac{4A'_+(x_+)A'_-(x_-)}{(A_+(x_+)+A_-(x_-))^2} \tag{1.2}$$

where the cone variables

$$x_\pm=x\pm t \qquad \partial_\pm=\tfrac{1}{2}(\partial_x\pm\partial_t) \tag{1.3}$$

are introduced. As in our previous works we are interested in N-singular solutions,

† Permanent address: Razmadze Mathematical Institute, Tbilisi, USSR.

i.e. the Cauchy data $\varphi(x)=\varphi(0,x)$ and $\pi(x)=\varphi_t(0,x)$ are correspondingly doubly and singly differentiable for any x with the exception of N points $x_1,\ldots,x_N$. In the neighbourhood of any point x_j there exist doubly and singly differentiable functions $f_j(x)$, $g_j(x)$ and real parameters $v_j(|v_j|<1)$ such that

$$\begin{aligned}\varphi(x)&=-\log\frac{(x-x_j)^2}{1-v_j^2}+f_j(x)\\ \pi(x)&=v_j\left(\frac{2}{x-x_j}+f_j'(x)\right)+g_j(x)\end{aligned}\tag{1.4}$$

with

$$f_j(x_j)=g_j(x_j)=0.$$

We require also definite asymptotics

$$\varphi(t,x)=-\log\tfrac{1}{4}[\omega(x-q_\pm)-pt]^2+\mathrm{O}(|x|^{-1}) \qquad x\to\pm\infty \qquad \omega=(p^2+4)^{1/2}\tag{1.5}$$

differentiable in x and t ($\partial_t\mathrm{O}(|x|^{-1})=\mathrm{O}(|x|^{-2})=\partial_x\mathrm{O}(|x|^{-1})$ and so on). In fact (1.5) fixes real parameters p (one for both ends) and $q_\pm$ which can be different at different ends. In Dzordzhadze *et al* (1979) and Pogrebkov and Polivanov (1985) requiring the preservation of (1.4) and (1.5) in dynamics we proved the existence and uniqueness theorem for the global solution of this Cauchy problem. This solution has N lines of singularities (corresponding to the zeros of the denominator of (1.2)) which are smooth, non-intersecting and timelike:

$$x=x_j(t)\in C^3 \qquad -1<\dot{x}_j(t)<1 \qquad j=1,\ldots,N.\tag{1.6}$$

In the vicinity of any line of singularity there exists a function $f_j(t,x)$ doubly differentiable in x and t such that

$$\varphi(t,x)=-\log\left(\frac{x-x_j(t)}{(1-\dot{x}_j^2(t))^{1/2}}\right)^2+f_j(t,x)\tag{1.7}$$

$$f_j(t,x_j(t))\equiv 0$$

and for the 'particle', i.e. for the point of singularity we have the equation of motion

$$\frac{\ddot{x}_j(t)}{1-(\dot{x}_j(t))^2}=f_{jx}(t,x_j(t)).\tag{1.8}$$

All this description is based on the following picture. Introduce two functions

$$U_\pm(t,x)=\left(\frac{\varphi_x\pm\varphi_t}{4}\right)^2-\left(\frac{\varphi_x\pm\varphi_t}{4}\right)_x+\frac{\mathrm{e}^\varphi}{4}\tag{1.9}$$

which are smooth, rapidly decreasing due to (1.4) and (1.5) and obey conservation laws

$$\partial_\mp U_\pm=0\tag{1.10}$$

due to (1.1). With the potentials $U_\pm(x)=U_\pm(0,x)$ we construct two pairs of solutions of two Schrödinger equations

$$-\chi''_{\pm i}(x)+U_\pm(x)\chi_{\pm i}(x)=0 \qquad \chi_{\pm1}\chi'_{\pm2}-\chi'_{\pm1}\chi_{\pm2}=1\tag{1.11}$$

obeying the additional condition

$$\chi_{+i}(x)=2(-1)^{i+j}\exp(-\varphi(x)/2)[\chi'_{-i}(x)+\tfrac{1}{4}(\varphi'(x)-\pi(x))\chi_{-i}(x)] \qquad i=1,2 \quad (1.12)$$

for $x_j<x<x_{j+1}$. Due to (1.9) this condition is consistent with (1.11). In terms of these solutions the Liouville functions in (1.2) are given as

$$A_{\pm}(x)=\frac{\chi_{\pm 2}(x)}{\chi_{\pm 1}(x)} \tag{1.13}$$

and the Liouville field as

$$2\exp(-\varphi(t,x)/2)=\chi_{+1}(x_+)\chi_{-2}(x_-)+\chi_{+2}(x_+)\chi_{-1}(x_-). \tag{1.14}$$

Note that the RHS has N zeros for any t which are just the singularities of φ. These zeros are of first order and we include the corresponding sign function into the exponential symbol in the LHS.

In addition we require that one of the potentials, say $U_+(x)$ has a solution of the problem (1.11) bounded on the whole axis, i.e. U_+ has a quasilevel. Choose χ_{+1} to be this solution. Then by (1.5) and (1.12)

$$\exists \lim_{x\to\pm\infty}\chi_{-1}(x)=\tfrac{1}{2}(\omega-p)\lim_{x\to\pm\infty}\chi_{+1}(x) \tag{1.15}$$

so the U_- potential also has the quasilevel. Then we proved that the number N of singularities of φ is given by

$$N=N_++N_-+1 \tag{1.16}$$

where $N_\pm$ are numbers of zeros of the functions $\chi_{\pm 1}(x)$.

In this way we have introduced a generalisation (1.14) of the Liouville solution (1.2) which enabled us to describe singular solutions in terms of regular functions $\chi_{\pm i}$. However the construction of Poisson bracket structure in this way meets serious obstacles. Now we propose a direct way based on the inverse scattering method and leading to self-consistent bracket structure for our singular solutions of the Liouville equation. The result is: a regular canonical d'Alembert field describing the continuous spectrum and a set of $2N$ canonical variables (discrete spectrum of the N-singular solution).

2. IST method for the Liouville equation

From the above discussion the exceptional role of the potentials $U_\pm(x)$ (1.9) is clear enough. These potentials are regular even for singular solutions, rapidly decrease at infinity and due to the Liouville equation (1.1) obey conservation laws (1.10). Now let the potentials $U_\pm(t,x)$ be of the form (1.9) where $\varphi(t,x)$ is an arbitrary real function. What are the equations for $\varphi(t,x)$ resulting from conservation laws (1.10)?

An easy exercise shows that (1.10) is equivalent to the following system

$$\begin{aligned}&[(\varphi_{tt}-\varphi_{xx}+2\,e^{\varphi})\,e^{-\varphi/2}]_x=0\\&\varphi_t(\varphi_{tt}-\varphi_{xx}+2\,e^{\varphi})=0\end{aligned} \tag{2.1}$$

which gives after integration

$$\varphi_{tt}-\varphi_{xx}+2\,e^{\varphi}=c(t)\,e^{\varphi/2} \tag{2.2a}$$

$$\varphi_t(t, x)c(t) = 0 \tag{2.2b}$$

where $c(t)$ depends only on t.

With smooth φ the function $c(t)$ is continuous. Therefore if there is no interval in t such that $\varphi_t(t, x) \equiv 0$ identically in x then from (2.2b) $c(t) \equiv 0$, and (2.2a) reduces to the Liouville equation. If there exist intervals of stationarity of $\varphi(t, x)$ then $c(t)$ is an arbitrary constant and (2.2a) gives

$$-\varphi_{xx} + 2\,\mathrm{e}^{\varphi} = c\,\mathrm{e}^{\varphi/2}.$$

This means that, in general, conservation laws (1.10) are not equivalent to the Liouville equation. Consider now our class of solutions described by the asymptotic condition (1.5) and singular behaviour (1.4). Let the functions $\varphi(t, x)$ from (1.9) obey the asymptotic condition (1.5). Then in the intervals of stationarity of $\varphi(t, x)$ the LHS of (1.2a) tends to zero faster than x^{-2} when $|x| \to \infty$. Meanwhile the RHS with c different from zero behaves as x^{-1}. This contradiction shows that with asymptotic behaviour (1.5) c have always to be equal to zero.

From the other side the character of the singularity (1.4) in the case of singular solutions also forbids non-zero c. Indeed, reconsidering once again (2.1) we see that $c(t)$ in (2.2) remains continuous except for discontinuity jumps at the points, where $\varphi(t, x)$ becomes singular. Then consider the behaviour of the LHS of (2.2a) to the left (right) of some singular point, when $\varphi_t(t, x) \equiv 0$, so that by (2.2b) c may be different from zero. Condition (1.4) shows that the LHS is regular when x tends to the point of singularity from the left (right), but the RHS behaves as $c|x|^{-1}$ so that again consistency requires $c = 0$.

Thus under required asymptotics (1.5) or singular behaviour (1.4) the conservation laws (1.10) are equivalent to the Liouville equation. However equations (1.10) are just consistency conditions for the following two systems

$$-y''_{\pm} + U_{\pm}y_{\pm} = k^2 y_{\pm} \tag{2.3}$$

$$\partial_{\mp} y_{\pm} = 0. \tag{2.4}$$

This consideration provides the most natural Lax pair for the (1.1):

$$L = \begin{pmatrix} -\partial_x^2 + U_+ & 0 \\ 0 & -\partial_x^2 + U_- \end{pmatrix} \qquad M = \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \tag{2.5}$$

and the condition

$$L_t = [L, M]$$

is equivalent to (1.10) and thus to (1.1).

Thus as in the standard IST scheme the analysis of the non-linear Liouville equation is reduced to the linear spectral problem (problems) (2.3). As our potentials $U_{\pm}$ are continuous and rapidly decreasing we introduce Jost functions

$$\begin{aligned} \varphi_{\pm}(x, k) &= \mathrm{e}^{-\mathrm{i}kx} + o(1) \qquad && x \to -\infty \\ \psi_{\pm}(x, k) &= \mathrm{e}^{-\mathrm{i}kx} + o(1) \qquad && x \to +\infty \end{aligned} \tag{2.6}$$

and transition matrices

$$\varphi_{\pm}(x, k) = a_{\pm}(k)\psi_{\pm}(x, k) + b_{\pm}(k)\bar{\psi}_{\pm}(x, k). \tag{2.7}$$

The functions $\varphi(x, k)$, $\psi(x, k)$, $a(k)$, $b(k)$ (we omit $\pm$ indices when describing properties common for both problems (2.3)) have the standard properties: $\varphi(x, k)$, $\bar{\psi}(x, k)$, $a(k)$ are analytic in $\operatorname{Im} k \geqslant 0$, $a(k)$ has a number of zeros in $\operatorname{Im} k > 0$, $\operatorname{Re} k = 0$, $\bar{a}(k) = a(-k)$, $\bar{b}(k) = b(-k)$, $\operatorname{Im} k = 0$,

$$|a(k)|^2 = 1 + |b(k)|^2 \tag{2.8}$$

and so on (see for example Zakharov *et al* 1980).

Now we have to write down the solution of the Liouville equation in terms of solutions of auxiliary linear problems (2.3). Note that (1.11) shows that we need zero-energy solutions of (2.3) bounded at both ends of the x axis. This means that the potentials $U_\pm$ possess quasilevels. Let us remember (see Calogero and Degasperis 1982) that in the general case the Jost function for $k = 0$ tending to one at one end linearly increases at the other end which corresponds to the pole behaviour of $a(k)$ and $b(k)$ at $k = 0$:

$$a_\pm(k) = \mathrm{i}c_\pm/k + \tilde{a}_\pm(k) \qquad b_\pm(k) = -\mathrm{i}c_\pm/k + \tilde{b}_\pm(k) \tag{2.9}$$

$c_\pm$ are real constants and $\tilde{a}$, $\tilde{b}$ are regular functions. Thus in our case we have additional conditions

$$c_\pm = 0 \tag{2.10}$$

and a, b are regular.

As a basis of zero-energy solutions of (1.3) we choose

$$\begin{aligned} &\psi_\pm(x) \equiv \psi_\pm(x, 0) \\ &\mathrm{i}\dot{\psi}_\pm(x) \equiv \mathrm{i}\frac{\partial}{\partial k}\psi_\pm(x, k)|_{k=0}. \end{aligned} \tag{2.11}$$

These functions are real, $\psi_\pm(x)$ are bounded on the whole x axis and $\mathrm{i}\dot{\psi}_\pm(x)$ linearly increase at both infinities,

$$\lim_{x\to+\infty} \psi_\pm(x) = 1 \qquad \mathrm{i}\dot{\psi}_\pm(x)/x = 1 + o(x^{-1}) \qquad x \to +\infty \tag{2.12}$$

and

$$W[\psi, \mathrm{i}\dot{\psi}] = 1. \tag{2.13}$$

We can choose '+' functions in (1.14) as

$$\chi_{+1}(x) = [\tfrac{1}{2}(\omega + p)]^{1/2}\psi_+(x) \qquad \chi_{+2}(x) = [\tfrac{1}{2}(\omega - p)]^{1/2}\mathrm{i}\dot{\psi}_+(x) \tag{2.14}$$

that automatically agrees with '+' conditions in (1.11). Then '−' functions must be determined by (1.12). However equivalently we may determine them by '−' conditions in (1.11) and asymptotic behaviour given by (1.12). From (1.15) at $+\infty$ we have

$$\chi_{-1}(x) = [\tfrac{1}{2}(\omega - p)]^{1/2}\psi_-(x) \tag{2.15}$$

and then

$$\chi_{-2}(x) = [\tfrac{1}{2}(\omega + p)]^{1/2}\mathrm{i}\dot{\psi}_-(x) - \text{constant} \times \psi_-(x).$$

The constant may be determined by asymptotic conditions (1.5), (2.12) which gives

$$\chi_{-2}(x) = [\tfrac{1}{2}(\omega + p)]^{1/2}\mathrm{i}\dot{\psi}_-(x) - [\tfrac{1}{2}(\omega - p)]^{1/2}\omega q_+\psi_-(x).$$

Finally the solution of the Liouville equation is given in the form

$$2\exp(-\varphi(t,x)/2)=\tfrac{1}{2}(\omega-p)\mathrm{i}\dot{\psi}_+(x_+)\psi_-(x_-)+\tfrac{1}{2}(\omega+p)\psi_+(x_+)\mathrm{i}\dot{\psi}_-(x_-) \\ -[\omega q-2\log(a(0)-b(0))^2]\psi_+(x_+)\psi_-(x_-) \tag{2.16}$$

where we introduced a new variable

$$q=q_+ +(2/\omega)\log(a(0)-b(0))^2 \qquad \omega=(p^2+4)^{1/2}. \tag{2.17}$$

Note that the condition (1.15) was exploited only at $+\infty$. At $-\infty$ by (2.14), (2.15) it gives

$$\lim_{x\to-\infty}|\psi_+(x)/\psi_-(x)|=1. \tag{2.18}$$

Now by (2.6)-(2.8) we see that

$$|a_+(0)-b_+(0)|=|a_-(0)-b_-(0)| \tag{2.19}$$

(note that $a_\pm(0)$ and $b_\pm(0)$ are real finite). Condition (2.19) is the reason for omitting subscripts '$\pm$' in (2.16) and (2.17) for $a(0)$ and $b(0)$.

Thus the scheme of solution of the Liouville equation is the following: from Cauchy data we construct potentials $U_\pm(x)$ and Jost solutions $\psi_\pm(x,k)$ of (2.3) and $a_\pm(k)$, $b_\pm(k)$. Then using the asymptotic condition (1.5) we determine p and q_+ and then q by (2.17). Substituting all these data into (2.16) we obtain the solution $\varphi(t,x)$ for arbitrary t.

Note that the RHS of (2.16) can become zero which generates singularities for $\varphi(t,x)$. These are zeros of first order and we remark that the corresponding sign function is included in the exponential in the LHS. The number of singularities of φ for arbitrary t is given by (1.15) where $N_\pm$ are now the numbers of zeros of $\psi_\pm(x)$. In the following we describe the Poisson bracket structure of the N-singular solutions in terms of scattering data. Note that in (2.16) we need Jost functions only at zero energy, but their reconstruction (by means of GLM equations, for example) needs the knowledge of scattering data for arbitrary k. Particularly the numbers $N_\pm$, playing a very important role in our analysis, are just the numbers of eigenstates for potentials $U_\pm(x)$ (see Calogero and Degasperis 1982). Denote by $\mathrm{i}\varkappa_j$ the zeros of $a(k)$:

$$a_\pm(\mathrm{i}\varkappa_{\pm j})=0 \qquad \varkappa_{\pm j}>0 \qquad j=1,\ldots,N_\pm \tag{2.20}$$

and by b_j corresponding coefficients

$$\varphi_\pm(x,\mathrm{i}\varkappa_{\pm j})=b_{\pm j}\bar{\psi}(x,\mathrm{i}\varkappa_{\pm j}) \tag{2.21}$$

which are real and have a property $b_j=(-1)^{N-j}|b_j|$ if $\varkappa_1<\ldots<\varkappa_N$ (cf Zakharov *et al* 1980).

Thus there are $2N_+ + 2N_-$ discrete variables emerging by IST from potentials $U_\pm$. From (2.16) it is seen that the Liouville solution is not fixed by these potentials only but also includes two additional discrete parameters p and q, thus we have precisely $2N$ (see (1.16)) discrete parameters for the N-singular solution.

Summarising, we obtain a description of the manifold of singular solutions of the Liouville equation in terms of potentials $U_\pm(x)$ and the pair (p,q). The potentials are subject to additional restrictions (2.10) and (2.19).

We now introduce the d'Alembert field $\phi(t, x)$ by means of Bäcklund transformation (see Lamb 1976):

$$\partial_{\pm}(\varphi \pm \phi) = \frac{\omega \pm p}{2} \exp \frac{\varphi \mp \phi}{2} \qquad \omega = (p^2+4)^{1/2}. \tag{2.22}$$

Substituting φ (1.16) we see that $\phi(t, x)$ may be chosen in the form

$$\exp(-\phi/2) = \psi_+(x_+)/\psi_-(x_-). \tag{2.23}$$

This ϕ is the real pseudoscalar free field, rapidly decreasing at both infinities together with its derivatives due to (2.10), (2.18). It has N_- negative ($\phi \to -\infty$) singularities and N_+ positive ones and again we include the corresponding sign factor in the exponential in the LHS of (2.23). This field does not depend on p, q and is completely determined by the $U_{\pm}(x)$ which are expressed in terms of ϕ as

$$U_{\pm} = \left(\frac{\phi_x \pm \phi_t}{4}\right)^2 \mp \left(\frac{\phi_x \pm \phi_t}{4}\right)_x. \tag{2.24}$$

Thus in fact we study by IST a free but singular d'Alembert field! In the simplest case when it is regular (i.e. $N_+ = N_- = 0$ or $N = 1$) all ingredients of (1.16) are expressible in terms of ϕ. Let

$$\phi(x) = \phi(0, x) \qquad \Pi(x) = \phi_t(0, x). \tag{2.25}$$

Then $\varphi(t, x)$ is given in Liouville form by

$$2\exp(-\varphi(t,x)/2) = \left(\tfrac{1}{2}(\omega - p)A_+(x_+) + \tfrac{1}{2}(\omega + p)A_-(x_-) - \omega q + 2\int \Pi(z)\,\mathrm{d}z\right)(A'_+(x_+)A'_-(x_-))^{-1/2} \tag{2.26}$$

where

$$A_{\pm}(x) = x + \int_x^{\infty} \mathrm{d}y \left[1 - \exp\left(\pm\tfrac{1}{2}\phi(y) - \tfrac{1}{2}\int_y^{\infty} \mathrm{d}z\,\Pi(z)\right)\right]. \tag{2.27}$$

(2.26) may be written in the form (2.16) if we notice that

$$\psi_{\pm}(x) = (A'_{\pm}(x))^{-1/2} = \exp\tfrac{1}{4}\left(\mp\phi(x) + \int_x^{\infty} \mathrm{d}\tilde{x}\,\Pi(\tilde{x})\right) \tag{2.28a}$$

$$\mathrm{i}\dot{\psi}_{\pm}(x) = \psi_{\pm}(x)A_{\pm}(x) \tag{2.28b}$$

$$\log(a(0) - b(0))^2 = \tfrac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}z\,\Pi(z). \tag{2.28c}$$

Formulae (2.28*a, b*) easily follow from (2.3) with $k = 0$ asymptotics (2.12) and (2.24), (2.25). As $\lim_{x\to-\infty}\psi_{\pm}(x) = a_{\pm}(0) - b_{\pm}(0)$ we see that the conditions $\phi \to 0$ when $x \to -\infty$ and $\int_{-\infty}^{+\infty} \mathrm{d}z\,\Pi(z)$ is finite are equivalent to (2.10), (2.19) and obtain (2.28*c*). Note that these explicit formulae work only in the case $N = 1$. If N_+ or $N_- \neq 0$ then these expressions become meaningless due to divergences. This is why we really need IST to resolve the general N-singular situation.

3. Poisson brackets for scattering data

Our aim is to write down a Poisson structure for the Liouville theory such that

$$\{\varphi_t(t,x), \varphi(t,y)\} = \delta(x-y) \tag{3.1}$$

with other brackets equal to zero. Because of the singularity of φ this problem is highly non-trivial. In order to be on the safe side we should write down brackets with exponentials $\exp(-\frac{1}{2}\varphi(t,x))$. The other problem is that for singular φ the definitions of $\delta\varphi$ and $\delta\pi$ $(\pi=\varphi_t)$ become ambiguous. Therefore we start with brackets for potentials $U_\pm$ which follow after formal use of (3.1) in (1.9):

$$\{U_\pm(x), U_\pm(y)\} = \pm\left(\frac{\delta'(x-y)}{4}(U_\pm(x)+U_\pm(y)) - \frac{\delta'''(x-y)}{8}\right) \tag{3.2a}$$

$$\{U_\pm(x), U_\mp(y)\} = 0. \tag{3.2b}$$

As $U_\pm$ are regular even for singular solutions we should postulate them for arbitrary N. Note that in fact we have additional conditions (2.10), (2.19) which are a kind of constraint and which are not taken into account up to now. Consider the case $N=1$ where these constraints follow automatically from the properties of regular free field $\phi(t,x)$. Note that in this case formulae (3.2) follow by (2.24) from canonical brackets for ϕ and Π:

$$\{\Pi(x), \phi(y)\} = \delta(x-y). \tag{3.3}$$

We define Poisson bracket of functionals of $\varphi(t,x)$ considering them as functionals of $\phi(x)$, $\Pi(x)$, q, p with use of (2.26) and (2.27) as

$$\{F,G\} = \{F,G\}_\phi + \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} \tag{3.4a}$$

where

$$\{F,G\}_\phi = \int \mathrm{d}x \left(\frac{\delta F}{\delta\Pi(x)}\frac{\delta G}{\delta\phi(x)} - \frac{\delta F}{\delta\phi(x)}\frac{\delta G}{\delta\Pi(x)}\right). \tag{3.4b}$$

Using the explicit formulae (2.27) and (2.28), we have

$$\begin{aligned}
&\frac{\delta\psi_\pm(y)}{\delta\phi(x)} = \pm\frac{\delta(x-y)}{4}\psi_\pm(y) \qquad \frac{\delta\psi_\pm(y)}{\delta\Pi(x)} = \tfrac{1}{4}\theta(x-y)\psi_\pm(y)\\
&\mathrm{i}\frac{\delta\dot\psi_\pm(y)}{\delta\phi(x)} = \pm\left(\frac{\delta(x-y)}{4}\mathrm{i}\dot\psi_\pm(y) + \frac{\theta(x-y)}{2\psi_\pm(y)}\right)\\
&\mathrm{i}\frac{\delta\dot\psi_\pm(y)}{\delta\Pi(x)} = \theta(x-y)\left(-\frac{\mathrm{i}}{4}\dot\psi_\pm(y) + \frac{\psi_\pm(y)}{2\psi_\pm(x)}\mathrm{i}\dot\psi_\pm(x)\right)\\
&\frac{\delta}{\delta\phi(x)}\log(a(0)-b(0))^2 = 0\\
&\frac{\delta}{\delta\Pi(x)}\log(a(0)-b(0))^2 = \tfrac{1}{2}.
\end{aligned} \tag{3.5}$$

Now straightforward calculation gives

$$\{\exp(-\tfrac{1}{2}\varphi(t_1,x_1)),\exp(-\tfrac{1}{2}\varphi(t_2,x_2))\}$$
$$=\tfrac{1}{32}[\varepsilon(x_{+1}-x_{+2})-\varepsilon(x_{-1}-x_{-2})]$$
$$\times\{2\exp[-\tfrac{1}{2}(\varphi(t_1,x_1)+\varphi(t_2,x_2))]+(\mathrm{i}\dot\psi_-(x_{-1})\psi_-(x_{-2})$$
$$-\mathrm{i}\psi_-(x_{-1})\dot\psi_-(x_{-2}))(\mathrm{i}\dot\psi_+(x_{+1})\psi_+(x_{+2})-\mathrm{i}\psi_+(x_{+1})\dot\psi_+(x_{+2}))\}. \tag{3.6}$$

This bracket is Poincaré invariant and annulates outside the light cone. Turning to equal time brackets we obtain canonical brackets (3.1) for the 1-singular Liouville field. Note that care is needed when using (3.5). Indeed if we substitute for F and G the functions $\psi_\pm(x)$ from (2.28a) we have finite results

$$\begin{aligned}&\{\psi_\pm(x),\psi_\pm(y)\}=\pm\tfrac{1}{16}\varepsilon(y-x)\psi_\pm(x)\psi_\pm(y)\\&\{\psi_\pm(x),\psi_\mp(y)\}=\mp\tfrac{1}{16}\psi_\pm(x)\psi_\mp(y)\end{aligned} \tag{3.7}$$

but if F and/or G are equal to $\mathrm{i}\dot\psi(x)$, then the brackets are infinite. In order to avoid these infinities in the calculation of (2.6) we have to take the variation of the RHS of (2.26) with the substitution of (2.27) and integrate over x in (3.4b) only in the last step.

In our previous work (Dzordzhadze *et al* 1979, Pogrebkov and Polivanov 1985) the modified traceless energy-momentum tensor was introduced. It differs from the canonical (Noether) one by some divergence term and is finite in the singular case. As the energy-momentum vector we used:

$$H_\phi=4\int\mathrm{d}x(U_+(x)+U_-(x))\qquad P_\phi=-4\int\mathrm{d}x(U_+(x)-U_-(x)) \tag{3.8}$$

obtained by integration from the modified tensor. In the same way for the Lorentz boost and dilatation generator we have:

$$M_\phi=4\int\mathrm{d}x\,(U_+(x)+U_-(x))x\qquad D_\phi=-4\int\mathrm{d}x\,(U_+(x)-U_-(x))x. \tag{3.9}$$

It is evident that in this way some constant terms can be lost. These terms are fixed by Poisson brackets. If we look for Hamiltonian (H), momentum (P), Lorentz (M) and dilatation (D) generators having correct brackets with Liouville field $\varphi(t,x)$ then in the one-singular case ($N=1$) we obtain

$$\begin{aligned}&H=H_\phi+\omega\qquad \omega=(p^2+4)^{1/2}\qquad P=P_\phi+p\\&M=M_\phi+\omega q\qquad D=D_\phi+pq.\end{aligned} \tag{3.10}$$

In this case after substituting (2.24) we have that generators (3.8), (3.9) are just the corresponding generators for the free regular field $\phi(t,x)$. Thus we have the correct field theory for $N=1$ and the next problem is to generalise it to arbitrary N.

In order to investigate this full problem we turn to IST results because the field ϕ becomes singular. Generalising to the $N=1$ case we consider p, q as canonical variables having zero Poisson brackets with potentials $U_\pm(x)$ and all their functionals. In other words we preserve two brackets (3.4) and our problem is to define correctly the bracket (3.5). In accordance with the above discussion we postulate (3.2). Then using the Leibnitz rule from (2.3) for the Jost function $\psi(x,k)$ defined by (2.6) we obtain

$$-\{\psi''(x,k),U(y)\}+U(x)\{\psi(x,k),U(y)\}=k^2\{\psi(x,k),U(y)\}-\{U(x),U(y)\}\psi(x,k).$$

Substituting (3.2) and using the Green function of the Sturm–Liouville equation in agreement with the asymptotic condition (2.6) we get

$$\{\psi_\pm(x,k), U_\pm(y)\}$$
$$=\pm\frac{\mathrm{i}k\theta(y-x)}{4\bar a_\pm(k)}\frac{\partial}{\partial y}\psi_\pm(y,k)(\psi_\pm(y,k)\bar\varphi_\pm(x,k)-\bar\varphi_\pm(y,k)\psi_\pm(x,k))$$
$$\pm\frac{\delta(y-x)}{4}\psi'_\pm(x,k)\pm\frac{\delta'(y-x)}{8}\psi_\pm(x,k).$$

Note that the condition (2.6) must be understood in the sense of distributions, in particular

$$\exp(\mathrm{i}kx)\xrightarrow[x\to\pm\infty]{}0\qquad \exp(\mathrm{i}kx)/k\xrightarrow[x\to\pm\infty]{}\pm\mathrm{i}\pi\delta(k). \tag{3.11}$$

Repeating this procedure we find

$$\{\psi_\pm(x,k), \psi_\pm(y,l)\}$$
$$=\frac{\mp 1}{8\mathrm{i}(l+k-\mathrm{i}0)}\Bigg(\mathrm{i}[k\theta(y-x)-l\theta(x-y)]\psi_\pm(x,k)\psi_\pm(y,l)$$
$$-\frac{l\theta(x-y)}{(l-k)\bar a_\pm(l)}W(\psi_\pm(x,l),\psi_\pm(x,k))(\psi_\pm(x,l)\bar\varphi_\pm(y,l)-\bar\varphi_\pm(x,l)\psi_\pm(y,l))$$
$$+\frac{k\theta(y-x)}{(l-k)\bar a_\pm(k)}W(\psi_\pm(y,l),\psi_\pm(y,k))$$
$$\times(\psi_\pm(y,k)\bar\varphi_\pm(x,k)-\bar\varphi_\pm(y,k)\psi_\pm(x,k))\Bigg) \tag{3.12a}$$
$$\{\psi_\pm(x,k),\psi_\mp(y,l)\}=0. \tag{3.12b}$$

It seems that we have now all the necessary ingredients to get the Poisson bracket by means of (2.16) and (2.11). Note, however, that the RHS of (3.12a) is the distribution of the type $(l+k-\mathrm{i}0)^{-1}$ thus we cannot put both $l=k=0$. If we try to do this we see that for example $\lim_{l,k\to 0}\{\psi(x,k),\psi(y,l)\}$ depends on the order of limits, limits including $\mathrm{i}\dot\psi$ do not exist in agreement with the case $N=1$ and from (3.12b) $\{\psi_+(x),\psi_-(y)\}=0$ which contradicts (2.7). To avoid all these difficulties we introduce Poisson brackets for scattering data. Considering now the limit of (3.12) when $x\to-\infty$ and using (2.6), (2.7) and (3.11) we come to

$$\{a_\pm(k),\psi_\pm(y,l)\}=\frac{\pm k}{8\mathrm{i}(l-k-\mathrm{i}0)}\left(\mathrm{i}a_\pm(k)\psi_\pm(y,l)-\frac{1}{l+k}W(\psi_\pm(y,l),\bar\psi_\pm(y,k))\varphi_\pm(y,k)\right)$$
$$\{b_\pm(k),\psi_\pm(y,l)\}=\frac{\mp k}{8\mathrm{i}(l+k-\mathrm{i}0)}\left(\mathrm{i}b_\pm(k)\psi_\pm(y,l)-\frac{1}{l-k}W(\psi_\pm(y,l),\psi_\pm(y,k))\varphi_\pm(y,k)\right). \tag{3.13}$$

In an analogous way when $y\to+\infty$ we get

$$\{a_\pm(k),a_\pm(l)\}=0$$
$$\{a_\pm(k),b_\pm(l)\}=\frac{\mp kla_\pm(k)b_\pm(l)}{4[(k+\mathrm{i}0)^2-l^2]} \tag{3.14}$$
$$\{b_\pm(k),b_\pm(l)\}=\mp\frac{\mathrm{i}\pi k}{4}\delta(k+l)a_\pm(k)a_\pm(l).$$

Using definitions of the parameters of discrete spectra (2.20), (2.21) we obtain from above

$$\begin{aligned}
&\{\varkappa_{\pm j}, a_\pm(k)\} = \{\varkappa_{\pm j}, b_\pm(k)\} = \{\varkappa_{\pm j}, \varkappa_{\pm l}\} = 0 \\
&\{\varkappa_{\pm i}, b_{\pm j}\} = \pm\tfrac{1}{8}\delta_{ij}\varkappa_{\pm i} b_{\pm j} \\
&\{a_\pm(k), b_{\pm j}\} = \mp\frac{\mathrm{i}k\varkappa_{\pm j}a_\pm(k)b_{\pm j}}{4(k^2+\varkappa_{\pm j}^2)} \\
&\{b_\pm(k), b_{\pm j}\} = \{b_{\pm i}, b_{\pm j}\} = 0 \qquad i, j = 1, \ldots, N_\pm
\end{aligned} \tag{3.15}$$

and in (3.12)-(3.15) all brackets of the mixed type as (3.12*b*) equal to zero.

Note that the initial bracket (3.2) is known as the second in the hierarchy of KdV brackets or Magri brackets (Kulish and Reiman 1978, Magri 1978). Therefore (3.14), (3.15) are very similar to brackets for KdV, in particular our bracket for $\{a(k), b(l)\}$ equals the respective one for KdV multiplied by k^2 (see Faddeev and Takhtadjan 1985). After this multiplication it seems that the residue c in (2.9) is an annulator in the bracket algebra (3.14). However it is necessary to be more careful. Indeed, by (3.14) and (2.9) we have

$$\{\mathrm{i}c + k\tilde{a}(k), b(l)\} = \frac{k}{4}\frac{(\mathrm{i}c + k\tilde{a}(k))(-\mathrm{i}c + lb(l))}{(k+\mathrm{i}0)^2 - l^2}$$

and note that

$$\frac{2k}{(k+\mathrm{i}0)^2 - l^2} = \frac{1}{k+l+\mathrm{i}0}) + \frac{1}{k-l+\mathrm{i}0} \xrightarrow[k\to 0]{} -2\pi\delta(l) \tag{3.16}$$

i.e. is not equal to zero! Thus

$$\{c, b(k)\} = -\tfrac{1}{4}\pi c^2\delta(k). \tag{3.17}$$

This means that calculations of brackets including c can be very ambiguous. Details will be given elsewhere, here we only list some results: c is an annulator for potentials $U_\pm(x)$

$$\{c, U(x)\} = 0 \tag{3.18}$$

and Jost functions

$$\{c, \psi(x, k)\} = \{c, \varphi(x, k)\} = 0 \qquad \text{for } k \neq 0 \tag{3.19}$$

and

$$\{c, a(k)\} = 0 \tag{3.20}$$

but

$$\left\{c_\pm, \frac{\psi_\pm(x, k)}{k}\right\} = \mp\frac{\pi c_\pm}{8\mathrm{i}}\,\delta(k)\psi_\pm(y, 0).$$

The zero bracket would be

$$\{c, \psi(x, k)/(k - \mathrm{i}0)\} = 0.$$

This formula gives

$$\{c_\pm, \psi_\pm(x, 0)\} = \frac{c_\pm}{8}\,\psi_\pm(x, 0).$$

These brackets are compatible with the equation

$$\varphi(x, k) = (\mathrm{i}c/k + \tilde{a}(k))\psi(x, k) + (-\mathrm{i}c/k + \tilde{b}(k))\bar{\psi}(x, k)$$

which follows from (2.7), (2.9), and results from the distribution nature in the momentum variables of the RHS of (3.12)-(3.14) and limiting prescriptions of (3.11) type. In particular the determinant formula (2.18) has to be checked in the form

$$k[|a(k)|^2 - |b(k)|^2 - 1] = 0$$

thus (3.17) and (3.20) do not lead to contradiction.

All the above discussion originates from the fact that we have constraints (2.10). On the constraints surface all these odd effects disappear. In the following we put $c_\pm \equiv 0$, or $a(k)$ and $b(k)$ regular. This agrees with Poisson structure.

Now we turn to condition (2.19) which is another constraint in our theory. We see that $a(0)$, $b(0)$ annulate brackets (3.14) and

$$\{a(0), U(x)\} = \{b(0), U(x)\} = 0 \tag{3.21}$$

so they are annulators for potentials. However they are not annulators for such non-local functionals of $U_\pm(x)$ as the field $\phi(t, x)$ which we need. This is demonstrated by (2.28c) for the case $N = 1$. This results from the fact that to construct $\phi(t, x)$ we need brackets involving $b(k)/k$ which again have to be considered as a distribution: $1/k \equiv \mathrm{VP}k^{-1}$. These brackets must be rederived from the bracket $\{\psi(x, k), \psi(x, l)/l\}$. In general we can expect that the division of (3.12) by l can lead to the appearance of an additional $\delta(l)$ term. Taking into account the asymptotic conditions (3.11) regularity of a and b we see that there are now additional terms, so instead of (3.14)

$$\begin{aligned} \{a_\pm(k), b_\pm(l)/l\} &= \mp \frac{k a_\pm(k) b_\pm(l)}{4[(k+\mathrm{i}0)^2 - l^2]}, \\ \{b_\pm(k)/k, b_\pm(l)/l\} &= \mp \frac{\mathrm{i}\pi}{4l} \delta(k+l) a_\pm(k) a_\pm(l) \end{aligned} \tag{3.22}$$

which also do not contain additional terms. This leads to

$$\{\log(a_\pm(0) - b_\pm(0))^2, b_\pm(k)/k\} = \mp \tfrac{1}{2}\mathrm{i}\pi\delta(k) a_\pm(0)$$

so $a(0)$, $b(0)$ do not annulate the bracket with $b(k)/k$. How to deal with condition (2.19) we will explain later in the appropriate place.

4. Poisson brackets for N-singular solutions

In order to make the Poisson structure more explicit we introduce a new field ρ in the following form

$$\rho(t, x) = \frac{-1}{\mathrm{i}\pi} \sum_{\varepsilon=\pm} \varepsilon(-1)^{N_\varepsilon} \int \frac{\mathrm{d}k}{k} \exp[2\mathrm{i}k(x + \varepsilon t)] b_\varepsilon(k) \left(\frac{\log|a_\varepsilon(k)|^2}{|a_\varepsilon(k)|^2 - 1}\right)^{1/2}. \tag{4.1}$$

This is real regular d'Alembert field with asymptotics

$$\rho(t, x) \xrightarrow[x \to +\infty]{} \mp \sum_{\varepsilon=\pm} \varepsilon(-1)^{N_\varepsilon} b_\varepsilon(0) \left(\frac{\log|a_\varepsilon(0)|^2}{|a_\varepsilon(0)|^2 - 1}\right). \tag{4.2}$$

Note, that under (2.19), i.e. on the constraint surface, $\rho(t, x)$ rapidly decreases at space infinity, as this condition with (2.8) is equivalent to

$$(-1)^{N_+}b_+(0)=(-1)^{N_-}b_-(0) \qquad (-1)^{N_+}a_+(0)=(-1)^{N_-}a_-(0). \tag{4.3}$$

The field $\rho(t,x)$ as follows from (3.22) is canonical:

$$\{\rho_t(t,x),\rho(t,y)\}=\delta(x-y). \tag{4.4}$$

Introduce now the variables p_j, q_j for discrete degrees of freedom:

$$\begin{aligned} p_j&=\begin{cases}16\varkappa_{+j} & j=1,\dots,N_+\\ -16\varkappa_{-(j-N_+)} & j=N_++1,\dots,N_++N_-\end{cases}\\ q_j&=\begin{cases}(8/p_j)\log|b_{+j}| & j=1,\dots,N_+,\\ -(8/p_j)\log|b_{-(j-N_+)}| & j=N_++1,\dots,N_++N_-.\end{cases}\end{aligned} \tag{4.5}$$

Then from (3.15), (3.22) it follows that they have zero brackets with field $\rho(t,x)$ and are canonical:

$$\{p_i,q_j\}=\delta_{ij} \qquad i,j=1,\dots,N-1. \tag{4.6}$$

Thus beginning with the Magri bracket (3.2) for $U_\pm$ we succeed in diagonalising the bracket structure in terms of the variables $\rho(x)$ $\rho_t(x)$ describing continuous spectrum and p_i, q_i $(i=1,\dots,N-1)$ corresponding to discrete spectrum. To prove self-consistency of this description we have now to go all the way back to reconstruct Magri brackets and brackets for Liouville fields beginning with (4.4) and (4.6).

Inverting (4.1) we reconstruct $b(k)$ and $a(k)$ in terms of new variables with the help of the standard dispersion relation:

$$\begin{aligned} a_\pm(k)&=\prod_{j=1}^{N_\pm}\frac{k-\mathrm{i}\varkappa_{\pm j}}{k+\mathrm{i}\varkappa_{\pm j}}\exp\left(\tfrac{1}{32}\int \mathrm{d}x\,\mathrm{d}y\,(\rho'(x)\pm\dot\rho(x))(\rho'(y)\pm\dot\rho(y))\exp(2\mathrm{i}k|x-y|)\right)\\ b_\pm(k)&=\mp(-1)^{N_\pm}\int \mathrm{d}z\,\exp(-2\mathrm{i}kz(\rho'(z)\pm\dot\rho(z)))\\ &\quad\times\left(\frac{\exp[\frac{1}{16}\int \mathrm{d}x\,\mathrm{d}y\,(\rho'(x)\pm\dot\rho(x))(\rho'(y)\pm\dot\rho(y))\cos 2k(x-y)]-1}{\int \mathrm{d}x\,\mathrm{d}y\,(\rho'(x)\pm\dot\rho(x))(\rho'(y)\pm\dot\rho(y))\cos 2k(x-y)}\right)^{1/2}. \end{aligned} \tag{4.7}$$

Brackets (4.4), (4.6) lead to (2.22) as it is easy to check. Note that we are automatically on the constraint surface (2.10) as the condition that a and b are regular at $k=0$ is equivalent to integrability of ρ', $\dot\rho$. As to the second constraint (2.19), from (4.7) it directly follows that this is the first type of constraint in Dirac terminology, i.e. it annulates all brackets in the weak sense. Condition (2.19) is equivalent to the asymptotic fall of the field $\rho(t,x)$. Thus in the following we should always consider rapidly decreasing $\rho(t,x)$ which enables us to introduce the definition of brackets in terms of standard variation derivatives. Thus we can legitimately introduce the bracket

$$\{F,G\}_U=\int \mathrm{d}x\left(\frac{\delta F}{\delta\dot\rho(x)}\frac{\delta G}{\delta\rho(x)}-\frac{\delta G}{\delta\dot\rho(x)}\frac{\delta F}{\delta\rho(x)}\right)+\sum_{j=1}^{N-1}\left(\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}-\frac{\partial G}{\partial p_j}\frac{\partial F}{\partial q_j}\right). \tag{4.8}$$

As we have said all brackets of the previous section follow from (4.8). Thus all those brackets starting with (3.12) are contained in (4.8) by (4.7) and standard GLM argument.

Let us prove that the singular field ϕ is canonical with (4.8). Definition (2.23) tells us that we need the zero momentum bracket of Jost functions, but as we have seen corresponding limits in (3.12) are ill defined. Thus we begin with (3.13). As the RHS is a distribution either in $k+l$ or in $k-l$ we can put $l=0$, which gives

$$
\begin{aligned}
&\{a_\pm(k), \phi(t,x)\}_U \\
&\quad = \frac{\mathrm{i}}{8k}\left(\frac{\partial}{\partial x}\varphi_\pm(x_\pm,k)\bar{\psi}_\pm(x_\pm,k) - 2\varphi_\pm(x_\pm,k)\bar{\psi}_\pm(x_\pm,k)\frac{\psi'_\pm(x_\pm)}{\psi_\pm(x_\pm)}\right) \\
&\left\{\frac{b_\pm(k)}{k}, \phi(t,x)\right\}_U \\
&\quad = \frac{\mathrm{i}}{8k^2}\left(\frac{\partial}{\partial x}\varphi_\pm(x_\pm,k)\psi_\pm(x_\pm,k) - 2\varphi_\pm(x_\pm,k)\psi_\pm(x_\pm,k)\frac{\psi'_\pm(x_\pm)}{\psi_\pm(x_\pm)}\right).
\end{aligned}
\tag{4.9}
$$

Evidently (4.9) are singular at the points where $\psi_\pm(x_\pm)=0$, i.e. at the singularities of $\phi(t,x)$. But these singularities do not interfere with the behaviour in k which is of interest to us. So by (4.1) we can compute $\{\rho(x), \phi(t,y)\}_U$, $\{\dot{\rho}(x), \phi(t,y)\}_U$, and turning to the definition of variables of the discrete spectrum in the same way we obtain brackets $\{p_j, \phi(t,y)\}_U$ $\{q_j, \phi(t,y)\}_U$. Due to (4.8)

$$\{\rho(x), \phi(t,y)\}_U = -\delta\phi(t,y)/\delta\dot{\rho}(x)$$

and so on. In this way we obtain all necessary ingredients of (4.8) for $\{\phi(t_2,x_1), \phi(t_2,x_2)\}_U$. They are regular in x and as a result after complicated but trivial computations we have

$$\{\phi(t_1,x_1), \phi(t_2,x_2)\}_U = \tfrac{1}{4}[\varepsilon(x_1-x_2+t_1-t_2) - \varepsilon(x_1-x_2-t_1+t_2)] \tag{4.10}$$

which is the canonical different time bracket for a free field. This bracket is standard when ϕ is regular.

The transformation from $\phi(x)$, $\Pi(x)$ to $\rho(x)$, $\dot{\rho}(x)$, p_j q_j is canonical and in the case of regular ϕ (see (3.5))

$$\{\cdot,\cdot\}_\phi = \{\cdot,\cdot\}_U. \tag{4.11}$$

In the presence of singularities the LHS needs to be defined. First of all it is necessary to introduce the variations with respect to singular variables. These are defined naturally by

$$\frac{\delta F}{\delta\Pi(x)} = \{F, \phi(x)\}_U \qquad \frac{\delta F}{\delta\phi(x)} = \{\Pi(x), F\}_U \tag{4.12}$$

which we substitute into (3.5). However, as an example (4.9) demonstrates that the integrand now becomes singular in the integration variable x. The singularities are of the pole type x^{-n} and we overcome this difficulty by the standard prescription for distributions:

$$\int \frac{f(x)}{x^n}\,\mathrm{d}x = \mathrm{VP}\int \frac{f(x) - \Sigma_{i=0}^{n-2} f^{(i)}(0)x^i/i!}{x^n}\,\mathrm{d}x \tag{4.13}$$

and $x^{-1} = \mathrm{VP}x^{-1}$. Again omitting direct but tedious computations we prove that under this definition

$$\{\dot{\rho}(x), \rho(y)\}_\phi = \delta(x-y) \qquad \{p_j, q_i\} = \delta_{ij} \tag{4.14}$$

and so on which proves (4.11) in the general case and gives the precise sense for the notion of canonical transformation in the singular case.

Now we may describe the phase space for the singular free field. It is given by the direct sum of phase space $(\dot\rho, \rho)$ and $(p_i, q_i)_{i=1}^{N-1}$. $\dot\rho$ and ρ are smooth rapidly decreasing functions, q_i are arbitrary real, p_j are real non-zero and mutually non-equal. We numbered discrete variables in such a way that

$$\begin{aligned} &p_j > 0 \qquad && j = 1, \ldots, N_+ \\ &p_j < 0 \qquad && j = N_+ + 1, \ldots, N_+ + N_-. \end{aligned} \tag{4.15}$$

From GLM equations it is easy to note that the case where some $p_j = 0$ in fact corresponds to the absence of respective degrees of freedom. The coincidence of some p_i, p_j can also be considered by the limiting procedure and leads to the dropping of both these degrees of freedom from the phase space. This demonstrates fermionic character of 'particles' corresponding to singularities (cf Pogrebkov and Polivanov 1985).

The brackets obtained are non-degenerate for given N_+, N_- because, as we have seen, $a(k)$, $b(k)$ and $\varkappa_j$, b_j are uniquely determined by $\dot\rho$, ρ, p_j, q_j. Thus the numbers N_+ N_- have the standard meaning of topological charges.

Now we turn to the Liouville field (2.16). The bracket for the Liouville theory is given by (3.4), where now (3.4*b*) is understood in the sense (4.11)-(4.13). With (4.12) we can easily check formulae (3.5) in this case. Using the integration prescription (4.13) we again obtain the same bracket as in (3.6). Thus the cannonical character of the Liouville field in the general N-singular case is proven. Remember that we have additional variables p, q $(-\infty < p, q < \infty)$ connected with asymptotic behaviour of $\varphi(t, x)$ which have to be added to the phase space which has now $N = N_+ + N_- + 1$ discrete degrees of freedom.

The last thing we have to do in order to complete the description of Liouville field theory is to rewrite the generators of Poincaré care and dilatation symmetries in terms of new variables.

Trace identities (Zakharov *et al* 1980) are

$$\int U_\pm(x)\,\mathrm{d}x = \frac{2}{\pi}\int \log|a_\pm(k)|\,\mathrm{d}k - 4\sum_{j=1}^{N_\pm} \varkappa_{\pm j}.$$

Then (3.8) by (4.5) and (4.7) gives

$$\begin{aligned} H_\phi &= \tfrac{1}{2}\int \mathrm{d}x\,(\dot\rho^2(x) + \rho'^2(x)) - \sum_{j=1}^{N-1} |p_j| \\ P_\phi &= -\int \mathrm{d}x\,\rho'(x)\dot\rho(x) + \sum_{j=1}^{N-1} p_j. \end{aligned} \tag{4.16}$$

It is easy to check that these generators lead to correct brackets for $\phi(t, x)$ and then (3.10) gives the same for the Liouville field $\varphi(t, x)$:

$$\partial_t\varphi(t, x) = \{H, \varphi(t, x)\} \qquad \partial_x\varphi(t, x) = -\{p, \varphi(t, x)\}. \tag{4.17}$$

To construct boost and dilatation generators by (3.9) we need $\int xU_\pm(x)\,\mathrm{d}x$. Note that differentiating (3.13) by (2.3) we have

$$\begin{aligned} \{a_\pm(k), U_\pm(x)\} &= \mp\frac{\mathrm{i}k}{4}\frac{\partial}{\partial x}\,\varphi_\pm(x, k)\bar\psi_\pm(x, k) \\ \left\{\frac{b_\pm(k)}{k}, U_\pm(x)\right\} &= \pm\frac{\mathrm{i}}{4}\frac{\partial}{\partial x}\,\varphi_\pm(x, k)\psi_\pm(x, k). \end{aligned} \tag{4.18}$$

Using asymptotics

$$\varphi(x,k)\bar{\psi}(x,k)\xrightarrow[x\to\pm\infty]{} a(k) \qquad \varphi(x,k)\psi(x,k)\xrightarrow[x\to\pm\infty]{} \pm b(k)$$

and

$$\mathrm{i}a'(k)=\int \mathrm{d}x(\varphi(x,k)\bar{\psi}(x,k)-a(k)) \qquad \mathrm{i}b'(k)=-\int \mathrm{d}x\,\varphi(x,k)\psi(x,k)$$

by (4.18) we have

$$\left\{a_\pm(k),4\int xU_\pm(x)\,\mathrm{d}x\right\}=\mp ka'_\pm(k) \qquad \left\{b_\pm(k),4\int xU_\pm(x)\,\mathrm{d}x\right\}=\mp kb'_\pm(k). \tag{4.19}$$

Now exploiting (4.1), (4.5) and (3.9) we obtain

$$\{\rho(x),M_\phi\}_U=-x\dot{\rho}(x) \qquad \{\dot{\rho}(x),M_\phi\}_U=-(x\rho'(x))'$$
$$\{\rho(x),D_\phi\}_U=x\rho'(x) \qquad \{\dot{\rho}(x),D_\phi\}_U=(x\dot{\rho}(x))'$$
$$\{p_j,M_\phi\}_U=-|p_j| \qquad \{q_j,M_\phi\}_U=q_j\operatorname{sgn}p_j$$
$$\{p_j,D_\phi\}_U=p_j \qquad \{q_j,D_\phi\}_U=-q_j.$$

As the bracket is non-degenerate we have

$$\begin{aligned} M_\phi&=\tfrac{1}{2}\int x[\dot{\rho}^2(x)+\rho'^2(x)]\,\mathrm{d}x-\sum_{j=1}^{N-1}q_j|p_j| \\ D_\phi&=-\int x\rho'(x)\dot{\rho}(x)\,\mathrm{d}x+\sum_{j=1}^{N-1}q_jp_j. \end{aligned} \tag{4.20}$$

Again with (3.10) we obtain correct brackets for the Liouville field and the correct algebra of generators. Note that the Hamiltonian (3.10) with (4.16) is unbounded from below which is a known problem for singular ($N>1$) Liouville field theory. For the $N=1$ case the mass gap exists which is of help in application to the theory of strings (Pogrebkov and Polivanov 1985).

5. N-solition solutions

Now when we have a general scheme for the Liouville field theory we can easily write down the general N-soliton solutions. For these solutions

$$b_\pm(k)=0 \tag{5.1}$$

i.e. $\rho(t,x)\equiv 0$, and they correspond to the lowest energy in respective sectors, labelled by topological charge N. In this case the Jost function $\psi(x,k)$ is given (Zakharov *et al* 1980) by

$$\psi(x,k)=\exp(-\mathrm{i}kx)\left(1+\sum_{j=1}^{N}\frac{\mathrm{i}\varkappa_j\gamma_j(x)}{k-\mathrm{i}\varkappa_j}\right) \tag{5.2}$$

(where the indices $\pm$ for ψ, N and γ are omitted) and column $\gamma=(\gamma_1,\ldots,\gamma_N)^T$ is a solution of the system

$$A(x)\gamma=2, \tag{5.3}$$

where the matrix $A(x)$ is

$$\begin{aligned}(A_\pm(x))_{ij} &= \delta_{ij}E_{\pm i}(x) + \frac{2\varkappa_{\pm j}}{\varkappa_{\pm i}+\varkappa_{\pm j}} \\ E_{\pm i}(x) &= (|b_{\pm i}|^{-1}\exp 2\varkappa_{\pm i}x)\prod_{l=1}^{N_\pm}\frac{|\varkappa_{\pm i}-\varkappa_{\pm l}|}{\varkappa_{\pm i}+\varkappa_{\pm l}}.\end{aligned} \tag{5.4}$$

From these equations in the usual way we get with $k\to 0$ in (5.2) the potentials

$$U_\pm(x) = -2\frac{\partial^2}{\partial x^2}\log\det A_\pm(x). \tag{5.5}$$

However, to construct the Liouville solution the expressions at $k=0$ are necessary. Taking into account (5.1) and (5.2) from (2.16) we have

$$\begin{aligned}&2\exp(-\tfrac{1}{2}\varphi(t,x)) \\ &\quad= [\omega(x-q)-pt]\left(1-\sum_{j=1}^{N_+}\gamma_{+j}(x_+)\right)\left(1-\sum_{j=1}^{N_-}\gamma_{-j}(x_-)\right) \\ &\qquad -\frac{\omega-p}{2}\sum_{j=1}^{N_+}\frac{\gamma_{+j}(x_+)}{\varkappa_{+j}}\left(1-\sum_{j=1}^{N_-}\gamma_{-j}(x_-)\right) \\ &\qquad -\frac{\omega+p}{2}\left(1-\sum_{j=1}^{N_+}\gamma_{+j}(x_+)\right)\sum_{j=1}^{N_-}\frac{\gamma_{-j}(x_-)}{\varkappa_{-j}}.\end{aligned} \tag{5.6}$$

Let $\mathfrak{a}(x)=\det A(x)$ and $\mathfrak{a}_j(x)$ is the determinant of $A(x)$ with substitution of $(2,\ldots,2)^T$ instead of the jth column. Then by (5.3)

$$\gamma_j = \mathfrak{a}_j/\mathfrak{a}.$$

Introduce $\mathfrak{b}(x)=\det B(x)$ where

$$(B_\pm(x))_{ij} = (A_\pm(x))_{ij} - 2 = \delta_{ij}E_{\pm i}(x) - \frac{2\varkappa_{\pm i}}{\varkappa_{\pm i}+\varkappa_{\pm j}}. \tag{5.7}$$

Now it is easy to see that

$$\left(1-\sum_{j=1}^{N}\gamma_j\right)\mathfrak{a} = \mathfrak{a}-\sum_{j=1}^{N}\mathfrak{a}_j = \mathfrak{b}. \tag{5.8}$$

Equation (5.3) may be written in the form

$$B\gamma = 2\left(1-\sum_{j=1}^{N}\gamma_j\right)$$

or by (5.7) and (5.8) after division of the ith row by $\varkappa_i$

$$\frac{E_i\gamma_i}{\varkappa_i} - \sum_{j=1}^{N}\frac{2\varkappa_j}{\varkappa_i+\varkappa_j}\frac{\gamma_j}{\varkappa_j} = 2\frac{\mathfrak{b}}{\varkappa_i\mathfrak{a}}. \tag{5.9}$$

Considering this as a system for $\gamma_i/\varkappa_i$ we note that in the LHS we have B^T. Thus

$$\frac{\gamma_i(x)}{\varkappa_i} = \frac{2}{\mathfrak{a}}\mathfrak{b}_i(x)$$

where $\mathfrak{b}_i$ is the determinant of the matrix B with substitution of $(\varkappa_1^{-1},\ldots,\varkappa_N^{-1})$ for the

ith row. Then (5.6) gives

$$2\mathfrak{a}_+(x_+)\mathfrak{a}_-(x_-)\exp(-\tfrac{1}{2}\varphi(t,x))$$
$$=[\omega(x-q)-pt]\mathfrak{b}_+(x_+)\mathfrak{b}_-(x_-)-(\omega-p)\sum_{j=1}^{N_+}\mathfrak{b}_{+j}(x_+)\mathfrak{b}_-(x_-) \tag{5.10}$$
$$-(\omega+p)\sum_{j=1}^{N_-}\mathfrak{b}_{-j}(x_-)\mathfrak{b}_+(x_+).$$

Now it is easy to see that the RHS of (5.10) is the determinant of the $N\times N$ matrix

$$\Omega(t,x)=\begin{pmatrix}\omega(x-q)-pt & z_+ & z_- \\ 1 & B_+(x_+) & 0 \\ 1 & 0 & B_-(x_-)\end{pmatrix} \tag{5.11}$$

where

$$z_\pm=\left(\frac{\omega\mp p}{\varkappa_{\pm 1}},\ldots,\frac{\omega\mp p}{\varkappa_{\pm N_\pm}}\right)$$

and 1 is the column of length N_+ (or N_-), 0 is the zero matrix $N_-\times N_+$ (or $N_+\times N_-$).
The Liouville equation can be written in the form

$$\varphi=\log 2\,\partial_+\partial_-\varphi.$$

Substituting in the RHS of (5.10) and (5.11) we have the final answer

$$\varphi(t,x)=\log(-2\,\partial_+\partial_-\log\det^2\Omega(t,x)).$$

6. Concluding remarks

The canonical structure problem for the Liouville theory was discused in various papers. We have already mentioned papers by Andreev (1976) d'Hoker and Jackiw (1982), Jackiw (1984) and Gervais and Neveu (1982, 1983, 1984), and noticed the difference between these works and our approach. To our knowledge the first three works are the only papers in the literature which deal with the field theory, i.e. the theory in the infinite volume. In the other works (Gervais and Neveu 1982, 1983, 1984, Curtright and Thorn 1982, Braaten *et al* 1982, 1983, Kihlberg 1983, Johanson *et al* 1984, Johanson and Marnelius 1984) the theory in a box with some periodicity conditions was considered. In the last mentioned papers an attempt was made to discuss singular solutions, but in fact only the 1-singular case (our $N_\pm=0$) was investigated. In these papers the IST method was not used and the construction was based on the Bäcklund transformation (2.22) and the Liouville solution in the form (1.14). As is now clear this possibility indeed exists only for the 1-singular solution. In our paper such a solution is also treated without references to the IST. However the general case needs this method.

Acknowledgments

We thank L D Faddeev, S P Novikov and L A Takhtadzian for numerous fruitful discussions.

References

Andreev V A 1976 *Theor. Math. Phys.* **29** 1027
Braaten E, Curtright T and Thorn C 1982 *Phys. Lett.* **118B** 115
—— 1983 *Ann. Phys., NY* **147** 365
Calogero F and Degasperis A 1982 *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations* vol 1 (Amsterdam: North-Holland)
Curtright T and Thorn C 1982 *Phys. Rev. Lett.* **48** 1309
D'Hoker E and Jackiw R 1982 *Phys. Rev.* D **26** 3517
Dzordzhadze (Jorjadze) G P, Pogrebkov A K and Polivanov M K 1979 *Theor. Math. Phys.* **40** 221
Faddeev L D and Takhtadjan L A 1985 *Lett. Math. Phys.* to be published
Gervais G L and Neveu A 1982a *Nucl. Phys.* B **199** 59
—— 1982b *Nucl. Phys.* B **209** 125
—— 1983 *Nucl. Phys.* B **224** 329
—— 1984 *Nucl. Phys.* B **238** 125
Jackiw R 1984 *Quantum Theory of Gravity. Essays in Honor of the 60th Birthday of Bryce DeWitt* ed S M Christensen (Bristol: Hilger) p 403
Johanson L, Kihlberg A and Marnelius R 1984 *Phys. Rev.* D **29** 2798
Johanson L and Marnelius R 1984 *Preprints ITP 84-13, 84-33*, Göteborg, Sweden
Kihlberg A 1983 *Phys. Rev.* D **27** 2542
Kulish P P and Reiman A G 1978 *Leningrad, LOMI Math. Notes* **77** 134 (in Russian)
Lamb G L 1976 *Lecture Notes in Mathematics* vol 515 (Berlin: Springer) p 69
Magri F 1978 *J. Math. Phys.* **19** 1156
Pogrebkov A K and Polivanov M K 1985 *Sov. Sci. Rev.* C to be published
Zakharov V E, Manakov S V, Novikov S P and Pitaevsky L P 1980 *Theory of Solitons: Inverse Scattering Transform Method* (Moscow: Nauka)